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FURTHER COMMENTS ON THE APPLICATION OF THE METHOD OF AVERAGING TO THE STUDY OF THE ROTATIONAL MOTIONS OF A TRIAXIAL RIGID BODY, PART 3

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FURTHER COMMENTS ON THE APPLICATION OF THE METHOD OF AVERAGING TO THE STUDY OF THE ROTATIONAL MOTIONS OF A TRIAXIAL RIGID BODY, PART 3

1. Introduction

In [J.R.,1970]*, we derived a set of differential equations, analogous to the Lagrange planetary equations, for use in studying the perturbations of the rotational motions of extended rigid bodies. These differential (variational) equations are expressed in terms of the perturbing torques and they are valid for either conservative or nonconservative torques. In [A.R.,1971], [F.R.,1972] and [M.R.,1973], these variational equations were applied to the case of a rapidly spinning triaxial body moving in an elliptic orbit, in which the orbital plane is regressing at a constant rate. The explicit differential equations obtained in this application were integrated by the method of averaging (described in [F.R.,1971]) to develop secular analytical expressions, which, to first-order in a small parameter, describe the complete space motions of the rigid body under the influence of nonresonant gravity-gradient perturbations.

In this report, the effects of aerodynamic torque on the rotational motion of an orbiting satellite will be studied, as another example of the application of the variational equations derived in [J.R.,1970] and the method of averaging discussed in [F.R.,1971]. As an artificial satellite moves through the

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*References to our earlier reports of June 16, 1970, February 19, 1971, August 2, 1971, February 21, 1972, and March 27, 1973 are indicated by [J.R.,1970], [F.R.,1971], [A.R.,1971], [F.R.,1972] and [M.R.,1973], respectively.

Earth's atmosphere, there is a drag force introduced by the interactions of the oncoming molecular flow with the satellite hull. This drag force, which is one of the most important of the perturbing forces influencing the rotational motion of the satellite about its center of mass, brings about secular variations in all of the variables, i.e., $\psi_H, \theta_H, \phi_H, \theta', \phi', h$. In particular, and unlike the gravity-gradient force, this drag force brings about secular variations in both the magnitude of the angular momentum vector $\underline{h}$ and the coning angle θ'.

Beletskii [1] assumes a plastic or perfectly inelastic impact of the molecules of the upper atmosphere with the satellite hull and neglects the effects due to the rotation of the Earth's atmosphere. He further assumes that the linear speed of the satellite hull with respect to the center of mass is very small compared to the speed of the center of mass itself. He then derives a symbolic expression for the aerodynamic torque $\underline{N}_A$ about the center of mass, for a triaxial body with a general exterior surface. His expression is valid through terms of first order in the magnitude of the angular velocity of the rotation of the satellite relative to the atmosphere. His assumptions will be used in this report in deriving an explicit expression for the aerodynamic torque in terms of a convenient cylindrical coordinate system. This general expression will reduce to the expression for a satellite with a surface of revolution given in [1].

The explicit expression for the torque for a satellite with an arbitrary shape involves surface integrals whose limits depend upon the instantaneous surface of attack. Although this torque expression is analytically intractable for the general case, it does become manageable if the satellite has a surface of revolution. For this reason, the considerations here are restricted to the study of the secular effects of aerodynamic torque on an orbiting, uniaxial satellite. Beletskii [1] uses the same torque expression* to study the secular motion of the angular momentum vector for a rapidly spinning, uniaxial, orbiting body with a surface of revolution by applying the method of averaging to the Euler's dynamical equations. In this report, however, the secular variations of all the six variables which describe the complete rotational motion of such a body about its center of mass under the influence of aerodynamic torque are studied by applying the method of averaging to the variational equations. Approximate, averaged, first-order, ordinary differential equations for the six variables are obtained, which are particularly convenient for numerical computation. If the problem is further simplified to the case of an uniaxial body moving in a circular orbit with constant air density, the approximate, averaged differential equations for the magnitude of angular momentum vector and coning angle can be integrated. It is found, as a verification of the results given

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*There is a misprint in the torque expression for a satellite with a surface of revolution given in [1] [equation (1.3.11 p. 16]. In a private communication, Beletskii agrees that the correct expression for the torque is given by equation (3.17), of this report.

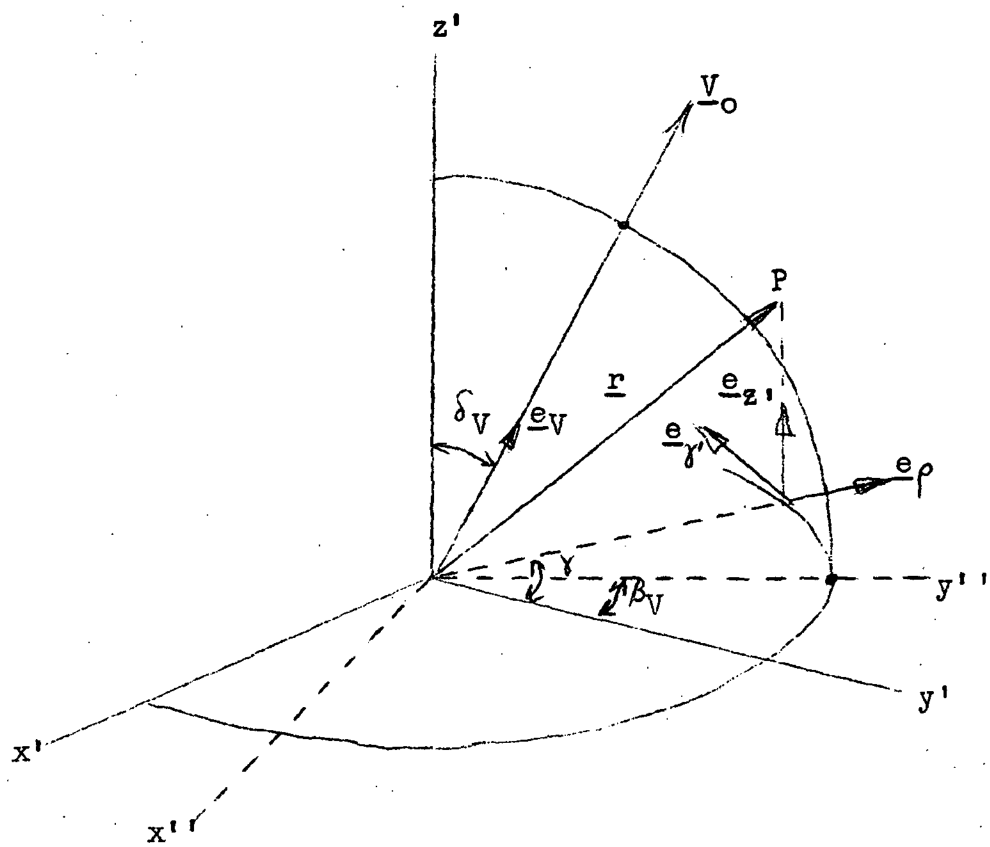


Figure 2.1

in [1], that the magnitude of the angular momentum vecter decreases exponentially with time approaching zero as a limiting value, while the coning angle may either decrease or increase with time approaching either zero or $\pi/2$ as limiting values, respectively.

2. Coordinate Systems

With reference to Figure 2.1, let $Ox'y'z'$ represent a body-fixed system whose origin O coincides with the center of mass of the satellite. The rectangular coordinate system $Ox''y''z''$ is obtained by rotating the system $Ox'y'z'$ about z' through the angle β_V in the sense shown. The angle β_V is determined by the condition that the y''-axis lies in the plane determined by the z'-axis and the velocity $\underline{V}_o$ (translated to O) of the center of mass with respect to an inertial frame at geocenter. The x''-axis is chosen to form a right-handed system. Let $\underline{i}'$, $\underline{j}'$, $\underline{k}'$ and $\underline{i}''$, $\underline{j}''$, $\underline{k}''$, be the unit vectors along the x', y', z' and x'', y'', z'' axes, respectively. The angle δ_V is drawn from the z'-axis to the line determined by $\underline{V}_o$. The cylindrical coordinates ρ, γ, z' of an arbitrary surface point P of the body is introduced where γ is reckoned from the y'-axis in a counterclockwise direction in the $x'y'$-plane. The unit vectors associated with the cylindrical coordinates ρ, γ, z' are designated by $\underline{e}_\rho$, $\underline{e}_\gamma$, $\underline{e}_{z'}$, respectively.

The following relations can be obtained readily

$$\underline{e}_V = s_{\delta_V} \underline{j}'' + c_{\delta_V} \underline{k}'', \quad \underline{e}_V = \frac{\underline{V}_o}{|\underline{V}_o|} \qquad \text{(a)}$$

$$\begin{bmatrix} \underline{i}'' \\ \underline{j}'' \\ \underline{k}'' \end{bmatrix} = \begin{bmatrix} c_{\beta_V} & s_{\beta_V} & 0 \\ -s_{\beta_V} & c_{\beta_V} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i}' \\ \underline{j}' \\ \underline{k}' \end{bmatrix} \qquad \text{(b)} \qquad (2.1)$$

$$\begin{bmatrix} \underline{e} \\ \underline{e} \\ e_{z'} \end{bmatrix} = \begin{bmatrix} -s_{\gamma} & c_{\gamma} & 0 \\ -c_{\gamma} & -s_{\gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i}' \\ \underline{j}' \\ \underline{k}' \end{bmatrix} \qquad \text{(c)}$$

where $\underline{e}_V$ represents the unit vector along $\underline{V}_o$.

3. Aerodynamic Torque

Assume that the impacts of the air molecules of the oncoming flow with the surface of the satellite are plastic or perfect inelastic and neglect the effects due to the rotation of atmosphere. Beletskii [1] has shown that the aerodynamic torque $\underline{N}_A$, with respect to the body-fixed system, through terms of first order in the magnitude of the angular velocity of rotation of the satellite relative to the atmosphere, is given by

$$\underline{N}_A = \frac{1}{2} c \rho_a V_o^2 \int_{S^*} (\underline{n} \cdot \underline{e}_V)(\underline{e}_V \times \underline{r}) \, dS \tag{3.1}$$

$$+ \frac{1}{2} c \rho_a V_o \int_{S^*} [(\underline{n} \cdot \underline{\omega} \times \underline{r})(\underline{e}_V \times \underline{r}) + (\underline{n} \cdot \underline{e}_V)(\underline{\omega} \times \underline{r}) \times \underline{r}] \, dS$$

where c is a coefficient. The quantities ρ_a, $\underline{n}$, $\underline{r}$, and $\underline{\omega}$ represent the atmospheric density, unit outward normal of the surface element dS of the satellite, the position vector of the surface element dS and the angular velocity vector of the satellite's rotation about its center of mass, respectively. The domain of the surface integral is indicated by S and it is defined by the inequality

$$\underline{V}_o \cdot \underline{n} \geqq 0 \tag{3.2}$$

The remainder of this section will be devoted to the development of an explicit expression for the torque (3.1) in terms of the cylindrical coordinates ρ, γ, and z'. The unit outward normal to the surface element dS can be written either in the component form

$$\underline{n} = n_\rho \, \underline{e}_\rho + \frac{1}{\rho} n_\gamma \, \underline{e}_\gamma + n_{z'} \, \underline{e}_{z'} \tag{3.3}$$

or

$$\underline{n} = n_{x'} \, \underline{i}' + n_{y'} \, \underline{j}' + n_{z'} \, \underline{k}' \tag{3.4}$$

Using equation (2.1(c)), we can write

$$n_{x'} = -\rho\,\sigma_n\,s_\gamma - \frac{1}{\rho}\,n_\gamma\,c_\gamma \qquad (a)$$

$$n_{y'} = \rho\sigma_n\,c_\gamma - \frac{1}{\rho}\,n_\gamma\,s_\gamma \qquad (b) \qquad (3.5)$$

where $n_\rho = \rho\sigma_n$.

Let $\phi(\rho, \gamma, z')$ be the relation which describes the surface of the satellite. It is found that

$$\nabla\phi = \phi_\rho\,\underline{e}_\rho + \frac{1}{\rho}\,\phi_\gamma\,\underline{e}_\gamma + \phi_{z'}\,\underline{e}_{z'} \qquad (3.6)$$

and that

$$\underline{n} = \frac{2\rho\,\phi_{\rho^2}\,\underline{e}_\rho + (1/\rho)\phi_\gamma\,\underline{e}_\gamma + \phi_{z'}\,\underline{e}_{z'}}{[4\rho^2\,\phi^2_{\rho^2} + (\rho)^{-2}\,\phi^2_\gamma + \phi^2_{z'}]^{1/2}} \qquad (3.7)$$

$$\phi_{\rho^2} = \frac{\partial\phi}{\partial(\rho^2)}, \quad \phi_\rho = \frac{\partial\phi}{\partial\rho}$$

$$\phi_\gamma = \frac{\partial\phi}{\partial\gamma'}, \quad \phi_{z'} = \frac{\partial\phi}{\partial z'} \qquad (3.8)$$

Comparing expressions (3.3) and (3.7), we have

$$n_\rho = \frac{2\rho\,\phi_{\theta'^2}}{[4\rho^2\,\phi^2_{\rho^2} + (\rho)^{-2}\,\phi^2_\gamma + \phi^2_{z'}]^{1/2}} \qquad (a)$$

$$n_\gamma = \frac{\phi_{\gamma'}}{[4\rho^2 \phi^2_{\rho^2} + (\rho)^{-2}\phi^2_\gamma + \phi^2_{z'}]^{1/2}} \qquad \text{(b)}$$

(3.9)

$$n_{z'} = \frac{\phi_{z'}}{[4\rho^2 \phi^2_{\rho^2} + (\rho)^{-2}\phi^2_\gamma + \phi^2_{z'}]^{1/2}} \qquad \text{(c)}$$

When referenced to the body-fixed system, the position vector of the surface element dS has the following form

$$\underline{r} = x' \, \underline{i}' + y' \, \underline{j}' + z' \, \underline{k}' \tag{3.10}$$

where

$$x' = -\rho s_\gamma \, , \qquad y' = \rho c_\gamma \tag{3.11}$$

Using equations (3.5),(3.9),(3.10),(3.11) and the equations of transformation (2.1), we can express (3.1) in the more explicit form

$$\begin{aligned} \underline{N}_A &= \tfrac{1}{2} c \rho_a V_o^2 (N_{A1x'} \, \underline{i}' + N_{A1y'} \, \underline{j}' + N_{A1z'} \, \underline{k}') \\ &+ \tfrac{1}{2} c \rho_a V_o [(J_{11}\omega_{x'} + J_{12}\omega_{y'} + J_{13}\omega_{z'}) \, \underline{i}' \\ &+ (J_{21}\omega_{x'} + J_{22}\omega_{y'} + J_{23}\omega_{z'}) \, \underline{j}' \\ &+ (J_{31}\omega_{x'} + J_{32}\omega_{y'} + J_{33}\omega_{z'}) \, \underline{k}'] \end{aligned} \tag{3.12}$$

where, $\omega_{x'}$, $\omega_{y'}$, and $\omega_{z'}$ are the rectangular components of $\underline{\omega}$ expressed in the Ox'y'z' system, and

$$N_{A1x'} = \int_{S^*} (z' s_{\delta_V} c_{\beta_V} - \rho c_{\delta_V} c_\gamma) n_e \, dS \qquad (a)$$

$$N_{A1y'} = \int_{S^*} (z' s_{\delta_V} s_{\beta_V} - \rho c_{\delta_V} s_\gamma) n_e \, dS \qquad (b)$$

$$N_{A1z'} = \int_{S^*} (\rho s_{\delta_V} c_{\beta_V} s_\gamma - \rho s_{\delta_V} s_{\beta_V} c_\gamma) n_e \, dS \qquad (c)$$

$$J_{11} = \int_{S^*} \Big\{ [(n_{z'} - z' \sigma_n) \rho c_\gamma + \frac{1}{\rho} z' n_\gamma s_\gamma](z' s_{\delta_V} c_{\beta_V} - \rho c_{\delta_V} c_\gamma) - n_e (z'^2 + \rho^2 c_\gamma^2) \Big\} dS \qquad (d)$$

$$J_{12} = \int_{S^*} \Big\{ [(n_{z'} - z' \sigma_n) \rho s_\gamma - \frac{1}{\rho} z' n_\gamma c_\gamma](z' s_{\delta_V} c_{\beta_V} - \rho c_{\delta_V} c_\gamma) - n_e \rho^2 s_\gamma c_\gamma \Big\} dS \qquad (e)(3.13)$$

$$J_{13} = \int_{S^*} [n_\gamma (z' s_{\delta_V} c_{\beta_V} - \rho c_{\delta_V} c_\gamma) - n_e z' \rho s_\gamma] dS \qquad (f)$$

$$J_{21} = \int_{S^*} \Big\{ [(n_{z'} - z' \sigma_n) \rho c_\gamma + \frac{1}{\rho'} z' n_\gamma s_\gamma](z' s_{\delta_V} s_{\beta_V} - \rho c_{\delta_V} s_\gamma) - n_e \rho^2 s_\gamma c_\gamma \Big\} dS \qquad (g)$$

$$J_{22} = \int_{S^*} \Big\{ [(n_{z'} - z'\sigma_n)\rho s_\gamma - \frac{1}{\rho} z' n_\gamma c_\gamma](z' s_{\delta_V} s_{\beta_V} - \rho c_{\delta_V} s_\gamma) - n_e(z'^2 + \rho^2 s_\gamma^2) \Big\} dS \qquad (h)$$

$$J_{23} = \int_{S^*} [n_\gamma (z' s_{\delta_V} s_{\beta_V} - \rho c_{\delta_V} s_\gamma) + n_e z' \rho c_\gamma] dS \qquad (i)$$

$$J_{31} = \int_{S^*} \Big\{ [(n_{z'} - z'\sigma_n)\rho c_\gamma + \frac{1}{\rho} z' n_\gamma s_\gamma](\rho s_{\delta_V} c_{\beta_V} s_\gamma - \rho s_{\delta_V} s_{\beta_V} c_\gamma) - n_e z' \rho s_\gamma \Big\} dS \qquad (j)$$

$$J_{32} = \int_{S^*} \Big\{ [(n_{z'} - z'\sigma_n)\rho s_\gamma - \frac{1}{\rho} z' n_\gamma c_\gamma](\rho s_{\delta_V} c_{\beta_V} s_\gamma - \rho s_{\delta_V} s_{\beta_V} c_\gamma) + n_e z' \rho c_\gamma \Big\} dS \qquad (k)$$

(3.13)

$$J_{33} = \int_{S^*} [n(\rho s_{\delta_V} c_{\beta_V} s_\gamma - \rho s_{\delta_V} s_{\beta_V} c_\gamma) - n_e \rho^2] dS \qquad (\ell)$$

$$n_e = s_{\delta_V} s_{\beta_V} (\rho \sigma_n s_\gamma + \frac{1}{\rho} n_\gamma c_\gamma) + s_{\delta_V} c_{\beta_V} (\rho \sigma_n c_\gamma - \frac{1}{\rho} n_\gamma s_\gamma) + n_{z'} c_{\delta_V} \qquad (m)$$

Equation (3.12) gives the aerodynamic torque $\underline{N}_A$ for a body with a general surface. Even for short periods of time it is observed that $\underline{N}_A$ is time dependent, principally through δ_V and β_V. For longer periods of time $\underline{V}_o$ and $\underline{\omega}$ must also be

considered to be time dependent vector quantities as they appear in $\underline{N}_A$. The expression $\underline{N}_A$ becomes simpler for a body with a surface of revolution with respect to z'-axis since $n_\gamma = 0$. Moreever, if the semi-constrained system Ox''y''z'' is chosen as the reference system, the simplications

$$\beta_V = 0, \qquad \omega_{x'} = \omega_{x''}$$
$$\int_{S^*} s_\gamma \, d\gamma = \int_{S^*} s_\gamma c_\gamma \, d\gamma = \int_{S^*} s_\gamma c_\gamma^2 \, d\gamma = 0 \tag{3.14}$$

may be introduced and then it is found that

$$N_{A1x'} = N_{A1y'} = J_{12} = J_{13} = J_{21} = J_{31} = 0 \qquad \text{(a)}$$

$$N_{A1z'} = \frac{1}{2} \bar{c} \, (\delta_V) \, \underline{e}_V \times \underline{k} \qquad \text{(b)} \quad (3.15$$

$$J_{11} = -I_5 = -c\,\delta_V \int_{S^*} [(z'^2 + \rho^2 c_\gamma^2) n_{z'} + (n_{z'} - \sigma_n z') \rho^2 c_\gamma^2]$$
$$-s\,\delta_V \int_{S^*} [(z'^2 + \rho^2 c_\gamma^2) \sigma_n \rho c_\gamma - z'(n_{z'} - \sigma_n z') \rho c_\gamma] dS \qquad \text{(c)}$$

$$J_{22} = -I_3 = -c\,\delta_V \int_{S^*} [(n_{z'} - z' \sigma_n) \rho^2 s_\gamma^2 + n_{z'} (z'^2 + \rho^2 s_\gamma^2)] dS$$
$$-s\,\delta_V \int_{S^*} (z'^2 + \rho^2 s_\gamma^2) \sigma_n \rho \, c_\gamma dS \qquad \text{(d)}$$

$$J_{23} = I_4 = s\delta_V \int_{S^*} z' \sigma_n \rho^2 c_\gamma^2 \, dS + c\delta_V \int_{S^*} n_{z'} \rho \, c_\gamma \, dS \quad (e)$$

$$J_{32} = I_2 = c\delta_V \int_{S^*} z' n_{z'} \rho \, c_\gamma \, dS + s\delta_V \int_{S^*} [\sigma_n z' \rho^2 c_\gamma^2$$

$$+ \rho^2 s_\gamma^2 (n_{z'} - z' \sigma_n)] \, dS \quad (f) \qquad (3.15)$$

$$J_{33} = -I_1 = c\delta_V \int_{S^*} n_{z'} \rho^2 dS + s\delta_V \int_{S^*} \sigma_n \rho^3 c_\gamma \, dS \quad (g)$$

where

$$\bar{c}(\delta_V) = c(W_1 \, c\delta_V + W_2 \, s\delta_V - W_3 \, c\delta_V \cot \delta_V) \quad (a)$$

$$W_1 = \int_{S^*} (z' \, n_{z'} - \sigma_n \rho^2 \, c_\gamma^2) \, dS \quad (b)$$

$$(3.16)$$

$$W_2 = \int_{S^*} z' \, \sigma_n \rho \, c_\gamma \, dS \quad (c)$$

$$W_3 = \int_{S^*} n_{z'} \rho \, c_\gamma \, dS \quad (d)$$

Then expression (3.12) becomes

$$\underline{N}_A = \frac{1}{2} \rho_a V_o^2 \, \bar{c}(\delta_V) \, \underline{e}_V \times \underline{k}'' + \frac{1}{2} c \rho_a V_o \underline{I} \qquad (3.17)$$

where

$$\underline{I} = -I_5\,\omega_{x''}\,\underline{i}'' + (-I_3\,\omega_{y''} + I_4\,\omega_{z''})\,\underline{j}'' + (I_2\,\omega_{y''} - I_1\,\omega_{z''})\,\underline{k}' \tag{3.18}$$

Expression (3.17) is exactly the same as given in [1].

The aerodynamic torque (3.17) can be analyzed in two separate parts. Thefirst part, which arises because the center of pressure does not coincide with the center of mass of the satellite, will be referred as the restoring torque $\underline{N}_{AR}$. The second part which introduces damping effects caused by the spin of the satellite with respect to its center of mass, will be reffered as the dissipative torque $\underline{N}_{AD}$. In the notation just described, (3.17) can be rewritten in the form

$$\underline{N}_A = \underline{N}_{AR} + \underline{N}_{AD} \tag{3.19}$$

where

$$\underline{N}_{AR} = \frac{1}{2}\,\rho_a\,V_o^2\,\bar{c}(\delta_V)\;\underline{e}_V \times k'' \qquad \text{(a)}$$

$$(3.20)$$

$$\underline{N}_{AD} = \frac{1}{2}\,c\,\rho_a\,V_o\,\underline{I} \qquad \text{(b)}$$

4. Approximations of the Aerodynamic Torque

Equation (3.12) gives a description of the aerodynamic torque for a body of arbitrary shape. In principle, the torque is to be calculated by integrating over the surface of attack S^*. It is noted, however, that the integrals which are involved may be analytically quite intractable since the integration

limits depend upon the surface of attack S^*. In turn, the surface of attack may depend discontinuously upon the time through the variables β_V and δ_V. Simplification may be achieved in certain important special cases, however. For example, in this report, the perturbation caused by aerodynamic torque will be analyzed in the special case where the satellite possesses a surface of revolution and the semi-constrained system is chosen to be the reference. In what follows, the considerations are restricted to this special case.

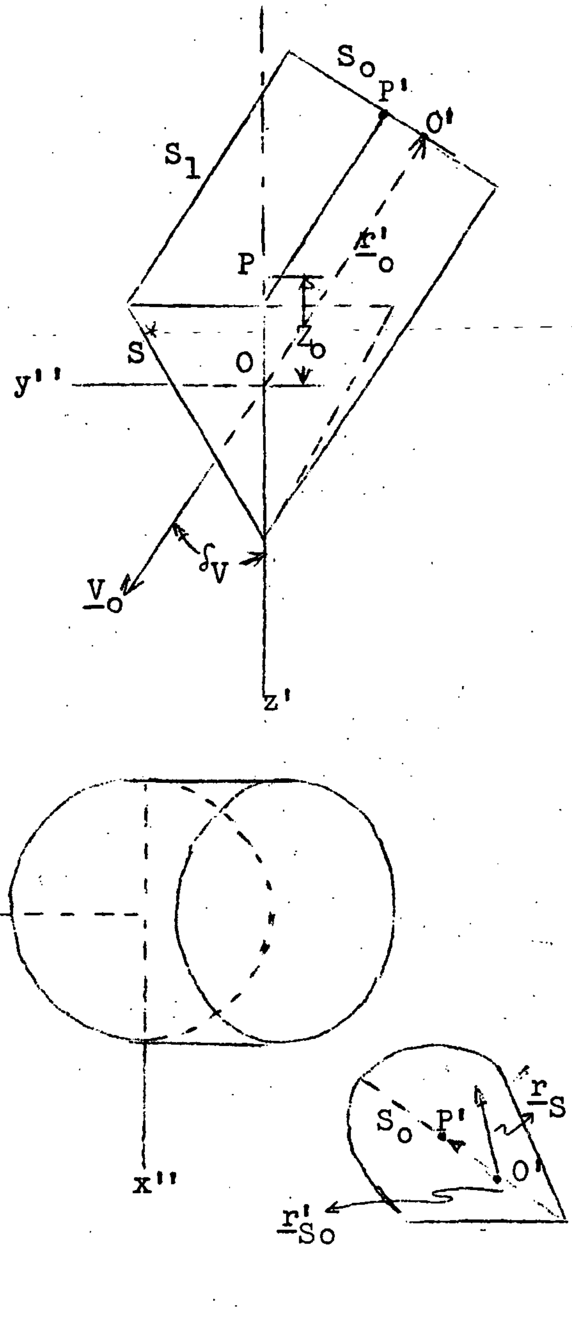
It is indicated by Beletskii[1] that the principal quantitative and qualitative effects of the aerodynamic torque which are common for various bodies of the type considered, can be described by using certain approximate formulas for W_i and I_j, where $i= 1,2,3$; $j=1,2,3,4,5$. These formulas are described briefly in this section.

(a) Restoring Torque

First of all, attention is concentrated on the restoring torque which is free of the spin of the satellite. Construct surface S_1, which is parallel to unit vector $\underline{e}_V$, and S_o, which is perpendicular to $\underline{e}_V$, in such a way that if combined with the surface of attack S^*, a closed surface in space is formed, as shown in Figure 4.1.

From equations (3.1) and (3.19), we have

$$\underline{N}_{AR} = \frac{1}{2} c \rho_a V_o^2 \underline{e}_V \times \int_{S^*} (\underline{n} \cdot \underline{e}_V) \underline{r} \, dS \qquad (4.1)$$



S_o
P'
O'
S_1
$\underline{r}'_o$
P
z_o
S
O
y''
δ_V
$\underline{V}_o$
z'
y''
x''
S_o
P'
O'
$\underline{r}_S$
$\underline{r}'_{So}$

Figure 4.1

Let $S^{*} + S_1 + S_o$ be a piecewise smooth orientable surface and let $\underline{u}$ be a vector function of the coordinates (x', y', z') which is continuous and has continuous first partial derivatives with respect to the coordinates in some domain τ containing $S^{*} + S_1 + S_o$. Using the divergence theorem, we may write

$$\int_{S^{*}+S_1+S_o} \underline{u}\cdot\underline{n}\, dS = \int_{\tau} \nabla\cdot \underline{u}\, d\tau \qquad (4.2)$$

and, since

$$(\underline{n}\cdot \underline{e}_V)\underline{r} = (x'\ \underline{e}_V\cdot\underline{n})\underline{i}' + (y'\ \underline{e}_V\cdot\underline{n})\underline{j}' + (z'\ \underline{e}_V\cdot\underline{n})k' \qquad (4.3)$$

we can write

$$\int_{S^{*}+S_1+S_o} (\underline{u}\cdot\underline{n})\underline{r}\, dS = \underline{i}'\int_{\tau} \nabla\cdot(x'\ \underline{e}_V)\, d\tau + \underline{j}'\int_{\tau} \nabla\cdot(y'\ \underline{e}_V)\, d\tau$$

$$+ \underline{k}'\int_{\tau} \nabla\cdot(z'\ \underline{e}_V)\, d\tau$$

Since $\underline{e}_V$ is independent of the coordinates x', y', and z', it can be shown that

$$\int_{S^{*}+S_1+S_o} (\underline{n}\cdot\underline{e}_V)\underline{r}\, dS = \tau\, \underline{e}_V \qquad (4.4)$$

where τ is the volume enclosed by the surface $S^{*} + S_1 + S_o$.

Since the surfaces S_1 and S_o are so constructed that

$$\underline{n}_1 \cdot \underline{e}_V = 0, \quad \underline{n}_o \cdot \underline{e}_V = -1 \tag{4.5}$$

the expression for the restoring aerodynamical torque takes the form

$$\underline{N}_{AR} = \frac{1}{2} c \rho_a V_o^2 \underline{e}_V \times \int_{S_o} \underline{r}' \, dS \tag{4.6}$$

where $\underline{r}'$ is the position vector of an arbitrary surface element of S_o with respect to the center of mass of the satellite.

Let O' be the projection of the center of mass on S_o, $\underline{r}'_o$ be the position vector of O', and $\underline{r}'_s$ be the position vector of any material point of S_o with respect to O'. We have, along surface S_o,

$$\underline{r}' = \underline{r}'_o + \underline{r}'_s \tag{4.7}$$

and hence

$$\underline{N}_{AR} = \frac{1}{2} c \rho_a V_o^2 S_o(\underline{e}_V \times \underline{r}'_{os}) \tag{4.8}$$

where $\underline{r}'_{os}$ is the position vector of the centroid of surface S_o with respect to O'. The vector product $\underline{e}_V \times \underline{r}'_{os}$ will be either in the positive or negative sense of x''. The magnitude of $\underline{N}_{AR}$ is

$$N_{AR} = \frac{1}{2} c \rho_a V_o^2 S_o r'_{os}, \quad r'_{os} = |\underline{r}'_{os}| \tag{4.9}$$

As shown in Figure 4.1, $r'_{os} = Z_o s_{\delta V}$, where Z_o is the

distance between the center of mass and the center of pressure, $Z_o = OP$. Let the projection of P on S_o be P', the centroid of S_o. Equation (4.8) can be rewritten in the form

$$\underline{N}_{AR} = \frac{1}{2} c \rho_a V_o^2 S_o Z_o (\underline{e}_V \times \underline{k}') \qquad (4.10)$$

Comparing (4.10) with expression (3.20(a)), it is found that

$$\bar{c}(\delta_V) = c S_o(\delta_V) Z_o(\delta_V) \qquad (4.11)$$

For perfect inelastic collisions, it is also true that

$$\bar{c}(\delta_V) = \bar{c}(\pi - \delta_V) \qquad (4.12)$$

Note, from equations (3.16(c)) and 3.16(d)), that, at an instant of time at which either $\delta_V = 0$ or $\delta_V = \pi$, $W_2 = W_3 = 0$. This suggests the approximations

$$W_2 = s_{\delta_V} f_2(\delta_V) \qquad (a)$$

$$(4.13)$$

$$W_3 = s_{\delta_V} f_3(\delta_V) \qquad (b)$$

Where f_2 and f_3 are functions of δ_V. Then equation (3.16(a)) becomes

$$\bar{c}(\delta_V) = c[W_1 c_{\delta_V} + f_2 - (f_2+f_3) c^2_{\delta_V}] \qquad (4.14)$$

which is a polynomial in c_{δ_V}. We may write (4.14) in the form

$$\bar{c}(\delta_V) = C_o + C_2 c^2_{\delta_V} \qquad (4.15)$$

where

$$C_o = c(W_1 \; c_{\delta_V} + f_2) \qquad \text{(a)}$$

$$(4.16)$$

$$C_2 = - \; c(f_2+f_3) \qquad \text{(b)}$$

In general C_o and C_2 depend slowly upon time. A meaningful simple approximation to (4.14) is obtained by assuming that both C_o and C_2 are constants which can be determined either by using equation (4.11) or any alternative method such that expressions (4.15) give reasonable approximation of $\bar{c}(\delta_V)$.

If the collisions between the molecules of the oncoming flow and the surface of the satellite are not perfectly inelastic and reflections occur, then equation (4.12) does not apply. In this circumstance, the simple approximation

$$\bar{c}(\delta_V) = C_o' + C_1 \; c_{\delta_V} + C_2 \; c^2_{\delta_V} \qquad (4.17)$$

where $C_o' = c \; f_2$, $C_1 = c \; W_1$, may be introduced. If however, the satellite is also symmetric about a plane which is perpendicular to the sxis of symmetry, equation (4.12) will hold irrespective of the reflections of the molecules and equation (4.15) will describe the case approximately. For a small angle of attack, we may even choose

$$\bar{c} = C_o' + C_1 + C_2 = \text{constant} \qquad (4.18)$$

(b) Dissipative Torque

Next consider the problem of approximating the dissipative part of the aerodynamic torque given by equation (3.20(b)).

Approximate formulas will be assumed for I_j, $j=1,2,3,4,5$. Recalling the reasoning for equations (4.13), it may be observed from equation (3.15) that, whenever the factor c_γ, appears in an integrand, the integral may be assumed in the form $s_{\delta_V} f_j(\delta_V)$. Therefore, we may consider that I_1, I_3 and I_5 are positive quantities for any value of δ_V and that the principal parts of these functions are constants. It is also noted that when either $\delta_V = 0$ or π, the difference $I_5 - I_3$ is zero. Thus, the principal parts of I_3 and I_5 may be approximated by the same constant. As to I_2 and I_4, they are functions of the form $s_{\delta_V} f_j(\delta_V)$. In summary, a simple approximate for $\underline{N}_{AD}$ may be obtained by assuming that

$$
\left.
\begin{aligned}
I_3 &= I_5 = C_{11} && \text{(a)}\\
I_1 &= C_{33} && \text{(b)}\\
I_2 &= C_{32} s_{\delta_V} && \text{(c)}\\
I_4 &= C_{23} s_{\delta_V} && \text{(d)}
\end{aligned}
\right\} \qquad (4.19)
$$

where the C_{ij} are constants. Reasonable estimates of these constants can be obtained by averaging the values of I_j at $\delta_V = 0$ and $\pi/2$.

5. First-Order, Secular Solutions for a Uniaxial Body with a Surface of Revolution Subjected to Restoring Aerodynamic Torque

Let the center of mass of the satellite move along an elliptic orbit about an attracting center at O. Assume that the inclination angle $\theta°$ of the orbital plane, the semimajor axis $\tilde{a}$, the eccentricity e and the rate of orbital precession $\dot{\Omega}$ are constants and let w represent the true anomaly. We can then write the well-known equation from orbital theory that

$$V_o = (\mu/p^*)^{1/2} f, \quad f = (1 + e^2 + 2e\, c_w)^{1/2} \qquad (5.1)$$

where μ is a constant equal to the product of the sum of the attracting mass and the satellite's mass with the gravitational constant and $p^* = a(1 - e^2)$. Let $0\, \bar{\xi}'\, \bar{\eta}'\, \bar{\zeta}'$ represent a rectangular coordinate system in which the $\bar{\zeta}'$ axis is perpendicular to the orbital plane, positive in the sense shown in Figure 5.1. The positive $\bar{\xi}'$ axis is chosen to coincide with the half line from 0 through perigee. Then $\bar{\eta}'$ is chosen to complete a right handed coordinate system. The unit vectors associated with the $0\, \bar{\xi}'\, \bar{\eta}'\, \bar{\zeta}'$ system are designated by $\underline{i}_{\bar{\xi}'}$, $\underline{j}_{\bar{\eta}'}$, $\underline{k}_{\bar{\zeta}'}$, respectively.
It is clear that

$$\underline{i}_{\bar{\xi}'} = c_w\, \underline{i}° + s_w\, \underline{j}° \qquad (a)$$

$$\underline{j}_{\bar{\eta}'} = s_w\, \underline{i}° + c_w\, \underline{j}° \qquad (b) \qquad (5.2)$$

$$\underline{e}_V = -\frac{1}{f}\, [s_w\, \underline{i}_{\bar{\xi}'} - (e + c_w)\, \underline{j}_{\bar{\eta}'}] \qquad (c)$$

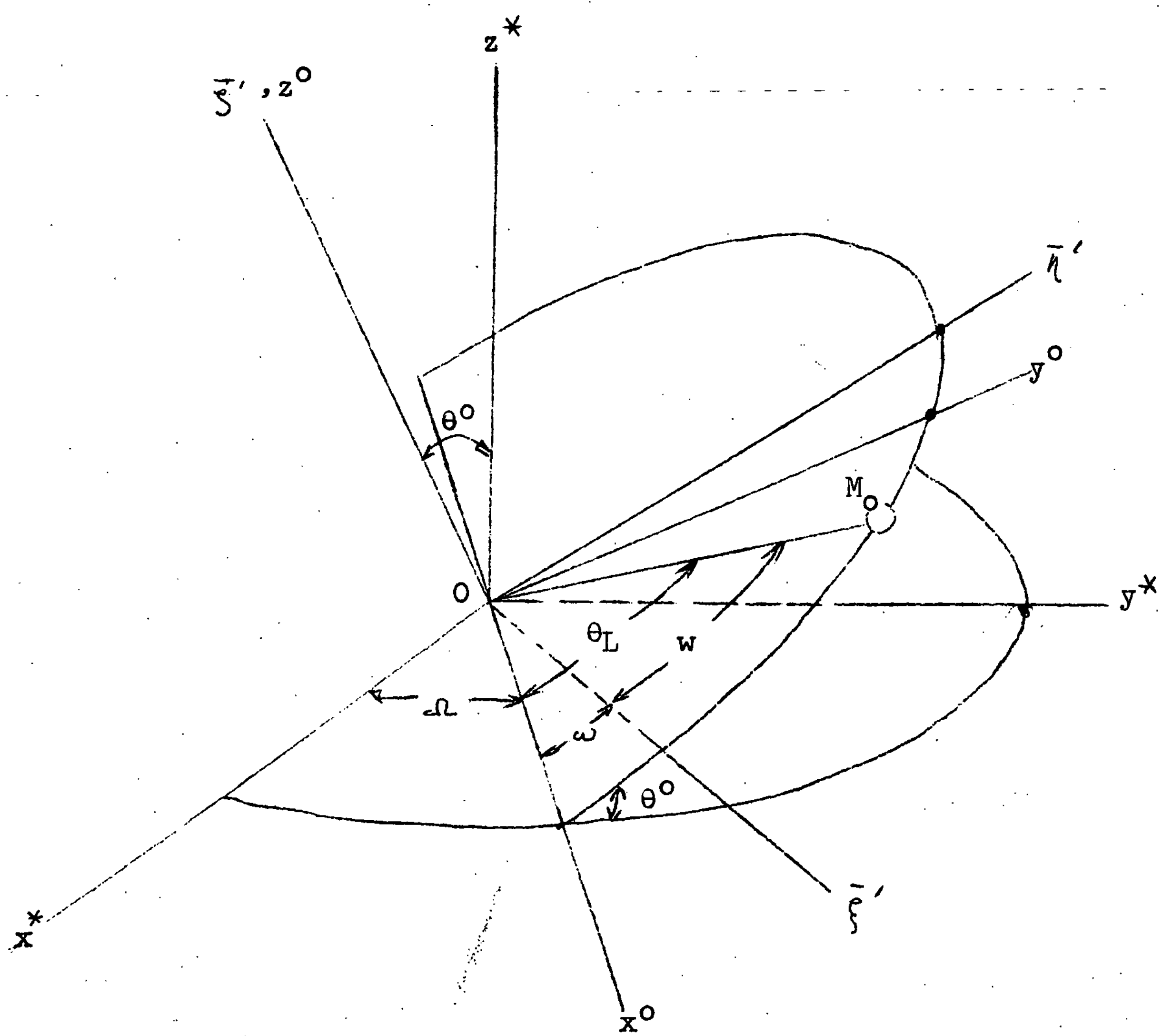


Figure 5.1

If the definitions

$$c_\Delta = c_{(\psi_H - \theta_L)} + e\, c_{(\psi_H - \omega)} \qquad \text{(a)}$$

$$(5.3)$$

$$s_\Delta = s_{(\psi_H - \theta_L)} + e\, s_{(\psi_H - \omega)} \qquad \text{(b)}$$

are introduced, it can be shown that

$$\underline{e}_V = \frac{1}{f}[(s_\Delta c_{\phi_H} + c_\Delta s_{\phi_H} c_{\theta_H})\underline{i} + (-s_\Delta s_{\phi_H} + c_\Delta c_{\phi_H} c_{\theta_H})\underline{j} - c_\Delta s_{\theta_H}\underline{k}\,] \quad \text{(a)}$$

$$\underline{e}_V \times \underline{k}' = \frac{1}{f}[(-s_\Delta s_{\phi_H} c_{\theta'} + c_\Delta c_{\phi_H} c_{\theta_H} c_{\theta'} - c_\Delta s_{\theta_H} s_{\theta'})\underline{i}$$

$$-(s_\Delta c_{\phi_H} c_{\theta'} + c_\Delta s_{\phi_H} c_{\theta_H} c_{\theta'})\underline{j} \qquad (5.4)$$

$$-(s_\Delta c_{\phi_H} s_{\theta'} + c_\Delta s_{\phi_H} c_{\theta_H} s_{\theta'})\underline{k}] \qquad \text{(b)}$$

$$c_{\delta_V} = \underline{e}_V \cdot \underline{k}' = \frac{1}{f}(s_\Delta s_{\phi_H} s_{\theta'} - c_\Delta c_{\phi_H} s_{\theta'} c_{\theta_H} - s_{\theta_H} c_{\theta'} c_\Delta) \qquad \text{(c)}$$

If the approximate formula (4.17) and the relations (5.1) through (5.4) are substituted into the torque equation (3.20(a)), it follows that

$$\underline{N}_{AR} = \underline{N}_{ARx}\,\underline{i} + N_{ARy}\,\underline{j} + N_{ARz}\,\underline{k} \qquad (5.5)$$

$$N_{ARx} = \frac{1}{2} f c \rho a \frac{\mu}{p^*} \Big\{ C_o' f(-s_\Delta s_{\phi_H} c_{\theta'} + c_\Delta c_{\phi_H} c_{\theta_H} c_{\theta'} - c_\Delta s_{\theta_H} s_{\theta'})$$

$$+ C_1 [(-s_\Delta^2 s_{\phi_H}^2 - c_\Delta^2 c_{\phi_H}^2 c_{\theta_H}^2 + \frac{1}{4} s_{2\Delta} s_{2\phi_H} c_{\theta_H}$$

$$+ \frac{1}{4} s_{2\Delta} s_{2\phi_H} c_{\theta_H} + c_\Delta^2 s_{\theta_H}^2) \; s_{\theta'} c_{\theta'}$$

$$+ \frac{1}{2} (2c_{\theta'}^2 - 1)(s_{2\Delta} s_{\phi_H} s_{\theta_H} - c_\Delta^2 c_{\theta_H} s_{2\theta_H})] +$$

$$+ \frac{C_2}{f} [(-s_\Delta^3 s_{\phi_H}^3 - s_\Delta c_\Delta^2 s_{\phi_H} c_{\phi_H}^2 c_{\theta_H}^2 + c_\Delta^3 c_{\phi_H}^3 c_{\theta_H}^3 \qquad (5.6)$$

$$+ s_\Delta s_{2\Delta} s_{\phi_H}^2 c_{\phi_H} c_{\theta_H} + c_\Delta s_\Delta^2 s_{\phi_H}^2 c_{\phi_H} c_{\theta_H}$$

$$- c_\Delta s_{2\Delta} s_{\phi_H} c_{\phi_H}^2 c_{\theta_H}^2 - 2c_\Delta^3 c_{\phi_H} s_{\theta_H}^2 c_{\theta_H}$$

$$+ c_\Delta s_{2\Delta} s_{\phi_H} s_{\theta_H}^2) \; s_{\theta'}^2 c_{\theta'}$$

$$+ (-\frac{1}{2} s_\Delta c_\Delta^2 s_{2\phi_H} s_{2\theta_H} + s_\Delta s_{2\Delta} s_{\phi_H}^2 s_{\theta_H} + 2c_\Delta^3 c_{\phi_H}^2 s_{\theta_H} c_{\theta_H}^2$$

$$- \frac{1}{4} c_\Delta s_{2\Delta} s_{2\phi_H} s_{2\theta_H} - c_\Delta^3 s_{\theta_H}^3) \; s_{\theta'} c_{\theta'}^2$$

$$+ (-c_\Delta s_\Delta^2 \; s_{\phi_H}^2 s_{\theta_H} - c_\Delta^3 \; c_{\phi_H}^2 s_{\theta_H} c_{\theta_H}^2$$

$$- \frac{1}{4} c_\Delta \; s_{2\Delta} s_{2\phi_H} s_{2\theta_H}) \; s_{\theta'}^3 \quad - - \; - - - -$$

$$+(- \; s_\Delta c_\Delta^2 \; s_{\phi_H} s_{\theta_H}^2 + c_\Delta^3 \; c_{\phi_H} s_{\theta_H}^2 c_{\theta_H}) \; c_{\theta'}^3]\Big\} \qquad \text{(a)} \; (5.6)$$

$$N_{ARy} = \frac{1}{2} fc \rho_a \frac{\mu}{p*} \Big\{ C_o' f(-s_\Delta c_{\phi_H} c_{\theta'} - c_\Delta s_{\phi_H} c_{\theta_H} c_{\theta'})$$

$$+ C_1 [(- \frac{1}{2} s_\Delta^2 \; s_{2\phi_H} + \frac{1}{2} s_{2\Delta} c_{\phi_H}^2 c_{\theta_H} - \frac{1}{2} \; s_{2\Delta} s_{\phi_H}^2 c_{\theta_H}$$

$$+ \frac{1}{2} c_\Delta^2 \; s_{2\phi_H} c_{\theta_H}^2) \; s_{\theta'} c_{\theta'}$$

$$- \frac{1}{2} (s_{2\Delta} \; c_{\phi_H} s_{\theta_H} + c_\Delta^2 \; s_{\phi_H} s_{2\theta_H}) \; c_{\theta'}^2]$$

$$+ \frac{C_2}{f} [(-s_\Delta^3 s_{\phi_H}^2 c_{\phi_H} - \frac{1}{2} \; c_\Delta s_{2\Delta} c_{\phi_H}^3 c_{\theta_H}^2 - \frac{1}{2} \; s_\Delta s_{2\Delta} \; s_{\phi_H}^3 \; c_{\theta_H}$$

$$+ s_\Delta s_{2\Delta} \; s_{\phi_H}^2 c_{\phi_H} c_{\theta_H} - c_\Delta^3 \; s_{\phi_H} c_{\phi_H}^2 c_{\theta_H}^3$$

$$- c_\Delta \; s_{2\Delta} \; s_{\phi_H}^2 c_{\phi_H} c_{\theta_H}^2) \; s_{\theta'}^2 \; c_{\theta'}$$

$$+ (-c_\Delta s_{2\Delta} c_{\phi_H}^2 s_{\theta_H} c_{\theta_H} + \frac{1}{2} \; s_\Delta s_{2\Delta} \; s_{2\phi_H} s_{\theta_H}$$

$$- c_\Delta^3 s_{2\phi_H} s_{\theta_H} c_{\theta_H}^2 + \frac{1}{2} c_\Delta s_{2\Delta} s_{\phi_H}^2 s_{2\theta_H}) s_{\theta'} c_{\theta'}^2$$

$$+(-\frac{1}{2} s_{2\Delta} c_\Delta c_{\phi_H} s_{\theta_H}^2 - c_\Delta^3 s_{\phi_H} s_{\theta_H} c_{\theta_H}) c_{\theta'}^3] \Big\} \qquad \text{(b) (5.6)}$$

- - - - - - - - - - - - - -

$$N_{ARz} = \frac{1}{2} f c \rho a \frac{\mu}{p^*} \Big\{ -C_o' f(s_\Delta c_{\phi_H} + c_\Delta s_{\phi_H} c_{\theta_H}) s_{\theta'}$$

$$+C_1 [-\frac{1}{2}(s_\Delta^2 s_{2\phi_H} - s_{2\Delta} c_{\phi_H}^2 c_{\theta_H} + s_{2\Delta} s_{\phi_H}^2 c_{\theta_H} - c_\Delta^2 s_{2\phi_H} c_{\theta_H}^2) s_{\theta'}^2$$

$$+ \frac{1}{2}(s_{2\Delta} c_{\phi_H} s_{\theta_H} + c_\Delta^2 s_{\phi_H} s_{2\theta_H}) s_{\theta'} c_{\theta'}$$

$$+ \frac{C_2}{f} [(-\frac{1}{2} s_\Delta^3 s_{\phi_H} s_{2\phi_H} - \frac{1}{2} s_{2\Delta} c_\Delta c_{\phi_H}^3 c_{\theta_H}^2 + s_\Delta s_{2\Delta} s_{\phi_H} c_{\phi_H} c_{\theta_H}$$

$$- \frac{1}{2} s_\Delta s_{2\Delta} s_{\phi_H}^3 c_{\theta_H} - c_\Delta^3 s_{\phi_H} c_{\phi_H}^2 c_{\theta_H}^3 + c_\Delta s_{2\Delta} s_{\phi_H}^2 c_{\phi_H} c_{\theta_H}^2) s_{\theta'}^3$$

$$+(-\frac{1}{2} c_\Delta s_{2\Delta} c_{\phi_H}^2 s_{2\theta_H} + \frac{1}{2} s_\Delta s_{2\Delta} s_{2\phi_H} s_{\theta_H} - c_\Delta^3 s_{2\phi_H} s_{\theta_H} c_{\theta_H}^2$$

$$+ \frac{1}{2} c_\Delta s_{2\Delta} s_{\phi_H}^2 s_{2\theta_H}) s_{\theta'}^2 c_{\theta'} + (-\frac{1}{2} c_\Delta s_{2\Delta} c_{\phi_H} s_{\theta_H}^2 +$$

$$- c_\Delta^3 s_{\phi_H} s_\theta^2 c_{\theta_H}) s_{\theta'} c_{\theta'}^2] \Big\} \qquad \text{(c) (5.6)}$$

When the torque expression (5.5) and relations (4.5)[‡] are substituted into equations (4.4)', equations (4.3) give the variational equations for the perturbed problem.

As in the analysis presented in [M.R.,1973], it is assumed that the precession rate $\dot{\Omega}$ and the aerodynamic torque are of the same order of magnitude. Let ϵ_1 represent the ratio of the magnitude of aerodynamic torque to the rotational energy of the orbiting body about its center of mass and define

$$X_i''(x_j;y_k) = \frac{1}{\epsilon_1} X_i(x_j;y_k), \quad i=1,2,\ldots,6 \qquad (5.7)$$

where j=1,2,...,6, k=1,2, and y_1, y_2 represent the fast variables $\bar{M}$, v^*, respectively. Then the variational equations (4.3) take the form

$$\dot{x}_i = \epsilon_1 X_i''(x_j;y_k), \quad i=1,2,\ldots,6 \qquad (5.8)$$

It is to be noted that the perturbing functions $X_i''(x_j;y_k)$, i=1, 2,...,6, are periodic functions of each y_k, k=1,2, with period 2π. Thus, the new dynamical system given by (5.8) and, (see [A.R.,1971,Eqs.(3.3(b)),(3.8)] and [F.R.,1972,Eq. (2.5(c))])*,

$$\dot{y}_1 = \dot{\bar{M}} = n \qquad \text{(a)} \quad (5.9)$$

---

‡ All equations designated (4.ij),i,j, nonnegative integers, refer to equations given in [F.R.,1972].

* When A = B, we have $\bar{k} = 0$, $q = 0$, $\beta = \pi/2$, $\gamma^2 = 0$, $\psi = 0$, $\Lambda_0(\pi/2,0) = 1$, $\Lambda_0(0,\bar{k}) = 0$, $\Omega_5 = u$, and $\Omega_1 = 0$.

$$\dot{y}_2 = \dot{u}^* = \dot{\phi}_H = \omega^* = \frac{h}{A} \qquad \text{(b)} \quad (5.9)$$

can be treated by the method of averaging.

A transformation

$$x_i = \bar{x}_i + \epsilon_1 R_i''(x_j; y_k), \quad i=1,2,\ldots,6 \qquad (5.10)$$

to new variables is introduced, so that, for suitable functions $R_i''(\bar{x}_j; y_k)$ the fast variables y_k, k=1,2, are eliminated from the transformed dynamical system to the first order in ϵ_1. If relations [F.R.,1971,Eq.(1.13(a))] are used, the transformed (the averaged) variational equations take the form

$$\dot{\bar{x}}_i = \epsilon_1 \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} X_i''(\bar{x}_j, y_k)\, d\bar{M}\, d\phi_H, \quad i=1,2,\ldots,6$$

or more simply

$$\dot{\bar{x}}_i = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} X_i(\bar{x}_j; y_k)\, d\bar{M}\, d\phi_H, \quad i=1,2,\ldots,6 \qquad (5.11)$$

The definitions

$$k_0 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a f^2 d\bar{M} \qquad \text{(a)}$$

$$k_1 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a \; s(\psi_H - \theta_L)\, f d\bar{M} \qquad \text{(b)} \quad (5.12)$$

$$k_2 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a c_{(\psi_H - \theta_L)} f d\bar{M} \qquad (c)$$

$$k_3 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a s^2_{(\psi_H - \theta_L)} f d\bar{M} \qquad (d)$$

$$k_4 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a s_{(\psi_H - \theta_L)} c_{(\psi_H - \theta_L)} f d\bar{M} \qquad (e)$$

$$k_5 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a c^2_{(\psi_H - \theta_L)} f d\bar{M} \qquad (f)$$

$$k_6 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a f^2 s_{(\psi_H - \theta_L)} d\bar{M} \qquad (g)$$

$$k_7 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a f^2 c_{(\psi_H - \theta_L)} d\bar{M} \qquad (h)$$

$$k_8 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a s^2_{(\psi_H - \theta_L)} d\dot{\bar{M}} \qquad (i) \quad (5.12)$$

$$k_9 = \frac{1}{2\pi} \int_0^{2\pi} \rho_a s_{(\psi_H - \theta_L)} c_{(\psi_H - \theta_L)} d\bar{M} \qquad (j)$$

$$k_{10} = \frac{1}{2\pi} \int_0^{2\pi} \rho_a \quad c^2_{(\psi_H - \theta_L)} \, d\bar{M} \tag{k}$$

$$k_{11} = \frac{1}{2\pi} \int_0^{2\pi} \rho_a \quad s^3_{(\psi_H - \theta_L)} \, d\bar{M} \tag{ℓ}$$

$$k_{12} = \frac{1}{2\pi} \int_0^{2\pi} \rho_a \quad s^2_{(\psi_H - \theta_L)} \, c_{(\psi_H - \theta_L)} \, d\bar{M} \tag{m}$$ (5.12)

$$k_{13} = \frac{1}{2\pi} \int_0^{2\pi} \rho_a \quad s_{(\psi_H - \theta_L)} \, c^2_{(\psi_H - \theta_L)} \, d\bar{M} \tag{n}$$

$$k_{14} = \frac{1}{2\pi} \int_0^{2\pi} \rho_a \quad c^3_{(\psi_H - \theta_L)} \, d\bar{M} \tag{o}$$

are introduced for economy of notation. After an integration over $\bar{M}$ and ϕ_H from 0 to 2π is perfomed, the averaged differential equations (5.11) take the form

$$\dot{\psi}_H = -(c_{\theta o}+s_{\theta o}\cot_{\theta_H} c_{\psi_H})\dot{\Omega}$$

$$+\frac{c\mu}{2hp^*}\Big\{C_o'[k_7+ek_o c_{(\psi_H-\omega)}]\cot_{\theta_H} c_{\theta'}$$

$$+\frac{C_1}{2}[k_5+ek_2 c_{(\psi_H-\omega)}]c_{\theta_H}(1-3c^2_{\theta'})$$

$$-C_2[\tfrac{1}{4}(k_{12}+ek_9 s_{(\psi_H-\omega)})+\tfrac{1}{2}(k_{12}+ek_8 c_{(\psi_H-\omega)})$$

$$+(k_{14}+ek_{10} c_{(\psi_H-\omega)})(2c^2_{\theta_H}-1)]\cot_{\theta_H} s^2_{\theta'} c_{\theta'}$$

$$+C_2(k_{14}+ek_{10} c_{(\psi_H-\omega)}) s_{\theta_H} c_{\theta_H} c^3_{\theta'}\Big\} \qquad \text{(a)} \quad (5.13)$$

$$\dot{\theta}_H = -s_{\theta o} s_{\psi_H}\dot{\Omega} - \frac{c\mu}{2hp^*}\Big\{-C_o'[k_6+ek_o s_{(\psi_H-\omega)}]s_{\theta'}$$

$$+\frac{C_1}{2}[2(k_4+ek_2 s_{(\psi_H-\omega)})s_{\theta_H}c^2_{\theta'}-(k_4+ek_1 c_{(\psi_H-\omega)})s_{\theta_H} s^2_{\theta'}$$

$$+e(k_2 s_{(\psi_H-\omega)}-k_1 c_{(\psi_H-\omega)})c_{\theta_H}\cot_{\theta_H} c^2_{\theta'}]$$

The variables $(\psi_H, \theta_H, \phi', \theta', h)$ will be used to represent the first order secular part of the variables in the remainder of this section

$$+ C_2[-\tfrac{1}{2}(k_{11}+ek_8 \quad (\psi_H-\omega)) - \tfrac{3}{2}(k_{13}+ek_{10}s_{(\psi_H-\omega)})c^2_{\theta_H}$$

$$+(k_{13}+ek_9 \quad c_{(\psi_H-\omega)})+\tfrac{1}{4}(k_{12}+ek_9 \quad s_{(\psi_H-\omega)})c_{\theta_H}]s_{\theta_H}s^2_{\theta'}c_{\theta'}$$

$$+ C_2[-\tfrac{1}{2}(k_{13}+ek_{10} \quad s_{(\psi_H-\omega)})c^3_{\theta'}$$

$$-(k_{14}+ek_{10} \quad c_{(\psi_H-\omega)})s_{\theta_H}s_{\theta'} \quad c^2_{\theta'}$$

$$-\tfrac{1}{2}(k_{13}+ek_{10} \quad s_{(\psi_H-\omega)})]s^2_{\theta_H}$$

$$- C_2 \quad ek_{10}(c_{(\psi_H-\omega)}-s_{(\psi_H-\omega)})c^2_{\theta_H}s^2_{\theta'} \quad c_{\theta'}\Big\} \qquad \text{(b)} \quad (5.13)$$

$$\dot{\phi}_H = \frac{h}{A}+\frac{s_{\theta^\circ} \quad c_{\theta_H}}{s_{\theta_H}} - \frac{c\,\mu}{2hp\kappa}\Big\{C'_o[k_7+ek_oc_{(\psi_H-\omega)}]c_{\theta_H}\cot_{\theta_H}c_{\theta'}$$

$$+\frac{C_1}{2}[(k_5+ek_2 \quad c_{(\psi_H-\omega)} \quad c^2_{\theta_H}(1-3 \quad c^2_{\theta'})$$

$$-(k_3+ek_1 \quad s_{(\psi_H-\omega)})c^2_{\theta'}+(k_5+ek_2 \quad c_{(\psi_H-\omega)})(2-3c^2_{\theta_H})c^2_{\theta'}$$

$$+ C_2[(k_{12}+ek_9 \quad s_{(\psi_H-\omega)})(\tfrac{1}{4}c_{\theta_H}\cot_{\theta_H}s^2_{\theta'}c_{\theta'}+ s_{\theta_H} \quad c^3_{\theta'})$$

$$+(k_{14}+ek_{10}\, c_{(\psi_H-\omega)})(2c^2_{\theta_H}-1)\,c^3_{\theta_H}\cot_{\theta_H}\, s^2_{\theta'}c_{\theta'}$$

$$-\frac{1}{2}\, s_{\theta_H}c^2_{\theta_H}s^2_{\theta'}c_{\theta'} + 2s_{\theta_H}\, c^2_{\theta_H}c^3_{\theta'} - 2s^3_{\theta_H}\, c^3_{\theta'}$$

$$+\frac{1}{2}(k_{12}+ek_8\, c_{(\psi_H-\omega)})(c_{\theta_H}\cot_{\theta_H}s^2_{\theta'}\, c_{\theta'}-s_{\theta_H}s^2_{\theta'}c_{\theta'})\,]\Big\} \quad (c) \quad (5.13)$$

$$\dot{\theta}' = \frac{ec\mu}{2hp^*}\Big\{\frac{C_1}{2}[k_2 s_{(\psi_H-\omega)}-k_1\, c_{(\psi_H-\omega)}]c_{\theta_H}s_{\theta'}c_{\theta'}$$

$$+C_2[c_{(\psi_H-\omega)}-s_{(\psi_H-\omega)}]k_{10}s_{\theta_H}c_{\theta_H}s_{\theta'}c^2_{\theta'}\Big\} \qquad (d)$$

$$\dot{\phi}' = -\frac{h}{A}\left(\frac{C-A}{C}\right)c_{\theta'} + \frac{c\mu}{2hp^*}\Big\{\frac{C_1}{2}[-(k_3+ek_1\, s_{(\psi_H-\omega)}]$$

$$+C_2[(k_{12}+ek_9 s_{(\psi_H-\omega)})\, s_{\theta_H}c^2_{\theta'} \; -\frac{1}{2}(k_{12}+ek_8 c_{(\psi_H-\omega)})\, s_{\theta_H}s^2_{\theta'}]$$

$$+\frac{C_1}{2}\,(k_5+ek_2\, c_{(\psi_H-\omega)})(2-3c^2_{\theta_H})\cdot c_{\theta'}$$

$$+C_2[(k_{14}+ek_{10}\, c_{(\psi_H-\omega)})(s_{\theta_H}c^2_{\theta_H}c^2_{\theta'}-\frac{1}{2}\, c^2_{\theta_H}s_{\theta_H}s^2_{\theta'} - 2s^3_{\theta_H}s^2_{\theta'}]\Big\} \quad (e)$$

$$\dot{h} = \frac{c\mu}{2hp^*}\Big\{\frac{C_1}{2}[k_2\, s_{(\psi_H-\omega)}-k_1\, c_{(\psi_H-\omega)}]c_{\theta_H}\, s^2_{\theta'}$$

$$+ C_2k_{10}\,[c_{(\psi_H-\omega)}-s_{(\psi_H-\omega)}]s_{\theta_H}c_{\theta_H}\, s^2_{\theta'}c_{\theta'}\Big\} \qquad (f)$$

Equations (5.13) represent the complete set of equations for determining the first order secular rotational motions of a uniaxial body under the influence of the restoring aerodynamic torque. The precession of the orbital plane and the regression of perigee are both considered since $\dot{\Omega}$ and $\dot{\omega}_o$ may have nonzero values. The above equations suggest that there are long-term secular changes in the angular momentum vector $\underline{h}$ as well as in the rest of the variables. The integrals $k_m, m=0,\ldots,14$, must be evaluated separately and they are functions of both ψ_H and ω and the atmospheric density ρ_a. In the following paragraph, the special case of a uniaxial body moving along a circular orbit will be examined.

In case of a circular orbit, the eccentricity e is zero. If we further assume that the molecular density of air ρ_a is a constant, it is found that the nonzero k_m's are

$$k_0 = \rho_a, k_3 = k_5 = k_8 = k_{10} = \frac{1}{2}\rho_a \qquad (5.14)$$

and the associated equations of motion are

$$\dot{\psi}_H = -(c_{\theta o}+s_{\theta o}\cot_{\theta_H} c\,\psi_H)\dot{\Omega} + \frac{c\rho_a V_o^2}{8h} C_1 c_{\theta_H}(1-3c^2_{\theta'}) \qquad (a)$$

$$\dot{\theta}_H = - s_{\theta o} s_{\theta_H}\dot{\Omega} \qquad (b)\ (5.15)$$

$$\dot{\phi}_H = \frac{h}{A} + \frac{s_{\theta o}\, c\psi_H}{s_{\theta_H}}\dot{\Omega} - \frac{c\rho_a V_o^2}{8h} C_1[c^2_{\theta'} + c^2_{\theta_H}(1-6c^2_{\theta'})] \qquad (c)$$

$$\dot{\theta}' = 0 \qquad \text{(d)}$$

$$\dot{\phi}' = -\frac{h}{A}\left(\frac{C-A}{C}\right)c_{\theta'} + \frac{c\,\rho_a V_o^2}{8h} C_1 (1 - 3c_{\theta_H}^2)\, c_{\theta'} \qquad \text{(e)} \quad (5.15)$$

$$\dot{h} = 0 \qquad \text{(f)}$$

where $V_o^2 = \mu / p^*$.

Equations (5.15) show that there is no secular variation in h and θ' to the first order, and if comparison is made between equations (5.15) and the associated averaged equations for perturbation due togravity-gradient torque given in [2], it can be seen that they differ only by a constant coefficient. In the case of ideally inelastic collisions between the air molecules and the surface of the satellite or if the satellite is also symmetric about a plane which is perpendicular to the axis of symmetry, we have $C_1 = 0$. Then it may be concluded that, in this special case, there is no secular change caused by the restoring aerodynamical torque.

6. First-Order, Averaged Differential Equations for a Uniaxial Body with a Surface of Revolution Subjected to Dissipative Aerodynamic Torque

The dissipative aerodynamic torque and an approximation to it have been derived in Sections 3 and 4 of this report. Suppose that the body possesses a surface of revolution with respect to the z'-axis and take the semi-constrained system as the reference system. We can write

$$\underline{N}_{AD} = \frac{1}{2} c \rho_a V_o [-C_{11} \omega_{x''} + (-C_{11} \omega_{y''} + C_{23}\, s_{\delta_V} \omega_{z''})\, \underline{j}''$$

$$+ (C_{32}\, s_{\delta_V} \omega_{y''} - C_{33}\, \omega_{z''}) \underline{k}''\ ] \qquad (6.1)$$

If the equations of transformation (2.1) are used, expression (6.1) becomes

$$\underline{N}_{AD} = \frac{1}{2}\, c \rho_a V_o [(-C_{11}\, \omega_{x'} - C_{23}\, s_{\delta_V} s_{\beta_V}\, \omega_{z'})\ \underline{i}'$$

$$+ (-C_{11}\, \omega_{y'} + C_{23}\, s_{\delta_V} c_{\beta_V}\, \omega_{z'})\ \underline{j}' \qquad (6.2)$$

$$+ (-C_{32} s_{\delta_V} s_{\beta_V} \omega_{x'} + C_{32} s_{\delta_V} c_{\beta_V} \omega_{y'} - C_{33}\, \omega_{z'}) \underline{k}'$$

The equations of transformation [A.R.,1971,Eq.(4.3(a))] can be used to transform (6.2) from the body-fixed system to the angular momentum system. The trigonometric functions of δ_V and β_V can be expressed in terms of the angles $\psi_H, \theta_L, \theta_H, \phi_H$, θ', ϕ' through equations (5.4(a)) and identities

$$s_{\delta_V}\, c_{\beta_V} = \underline{i}' \cdot (\underline{e}_V \times \underline{k}') \qquad (a)$$

$$(6.3)$$

$$s_{\delta_V}\, s_{\beta_V} \underline{k}' = \underline{i}' \times (\underline{e}_V \times \underline{k}') \qquad (b)$$

If then relations [A.R.,1971,Eqs.(4.3(a)),(4.4(a))], (5.3) and (5.4) are used, it is found that

$$s_{\delta_V} c_{\beta_V} = \frac{1}{f}[s_\Delta(-c_{\phi_H} s_{\phi'} - s_{\phi_H} c_{\phi'} c_{\theta'})$$

$$+c_\Delta(c_{\phi_H} c_{\theta_H} c_{\phi'} c_{\theta'} - s_{\theta_H} c_{\phi'} s_{\theta'} - s_{\phi_H} c_{\theta_H} s_{\phi'})] \qquad \text{(a)}$$

$$s_{\delta_V} s_{\beta_V} = \frac{1}{f}[s_\Delta(-c_{\phi_H} c_{\phi'} + s_{\phi_H} s_{\phi'} c_{\theta'}) \qquad (6.4)$$

$$+c_\Delta(-c_{\phi_H} c_{\theta_H} s_{\phi'} c_{\theta'} + s_{\theta_H} s_{\phi'} s_{\theta'} - s_{\phi_H} c_{\theta_H} s_{\phi'})] \qquad \text{(b)}$$

The components of $\underline{h}$ in the body-fixed system are

$$h_{x'} = h_\phi s_{\phi'} s_{\theta'} \qquad \text{(a)}$$

$$h_{y'} = h c_{\phi'} s_{\theta'} \qquad \text{(b)} \qquad (6.5)$$

$$h_{z'} = h c_{\theta'} \qquad \text{(c)}$$

It follows that, for a uniaxial body,

$$\omega_{x'} = \frac{h}{A} s_{\phi'} s_{\theta'} \qquad \text{(a)}$$

$$\omega_{y'} = \frac{h}{A} c_{\phi'} s_{\theta'} \qquad \text{(b)} \qquad (6.6)$$

$$\omega_{z'} = \frac{h}{C} c_{\theta'} \qquad \text{(c)}$$

since $h_{x'} = A\,\omega_{x'}$, $h_{y'} = A\,\omega_{y'}$, $h_{z'} = C\,\omega_{z'}$.

Through the combined use of [A.R.,1971,Eq.(4.3(a))], (6.4) and (6.6), equation (6.2) can now be expressed in the form

$$\underline{N}_{AD} = \underline{N}_{ADx}\underline{i} + N_{ADy}\,\underline{j} + N_{ADz}\,\underline{k} \tag{6.7}$$

where

$$N_{ADx} = \frac{1}{2C}\, c\,\rho_a V_o C_{23}\frac{1}{f}(s_\Delta c_{\phi_H} + c_\Delta s_{\phi_H} c_{\theta_H} c_{\theta'}) \qquad \text{(a)}$$

$$N_{ADy} = \frac{1}{2}\, c\,\rho_a V_o h\Big\{\frac{1}{A}[-C_{11}s_{\phi'}c_{\theta'} + \frac{C_{32}}{f}(s_\Delta s_{\phi_H} s^2_{\theta'}\, c_{\theta'}$$

$$+ c_\Delta(-c_{\phi_H} c_{\theta_H} s^2_{\theta'} c_{\theta'} + s_{\theta_H}\, s^3_{\theta'}))]$$

$$- \frac{1}{C}[\frac{C_{23}}{f}\, s_\Delta s_{\phi_H}\, c^3_{\theta'} - C_{33}s_{\theta'}\, c_{\theta'}$$

$$+ \frac{C_{23}}{f}\, c_\Delta(s_{\theta_H} s_{\theta'}\, c^2_{\theta'} - c_{\phi_H} c_{\theta_H} c^3_{\theta'})]\Big\} \qquad \text{(b)} \qquad (6.8)$$

$$N_{ADz} = \frac{1}{2}\, c\,\rho_a V_o h\Big\{\frac{1}{A}[-C_{11}\, s^2_{\theta'} - \frac{C_{32}}{f}(s_\Delta s_{\phi_H} s_{\theta'}\, c^2_{\theta'}$$

$$+ c_\Delta(-c_{\phi_H} c_{\theta_H} s_{\theta'}\, c^2_{\theta'} + s_{\theta_H}\, s^2_{\theta'} c_{\theta'}))]$$

$$- \frac{1}{C}[\frac{C_{23}}{f}\, s_\Delta\, s_{\phi_H} s_{\theta'}\, c^2_{\theta'} + C_{33}\, c^2_{\theta'}$$

$$+ \frac{C_{23}}{f}\, c_\Delta(s_{\theta_H} s^2_{\theta'}\, c_{\theta'} - c_{\phi_H} c_{\theta_H}\, s_{\theta'}\, e^2_{\theta'})]\Big\} \qquad \text{(c)}$$

It is to be noted that the dissipative torque $\underline{N}_{AD}$ is a continuous function of the slow variables $x_i, i=1,\ldots,6,$ and the fast variables $\bar{M}$ and ϕ_H and it is also a periodic function

of each of $\bar{M}$ and ϕ_H with a period of 2π. If the torque expression (6.7) and relations (5.5) are substituted into [F.R.,1972,Eqs.(4.4)], [F.R.,1972,Eqs.(4.3)] gives the variational equations for the perturbed problem. Again, the equations of transformation (5.10) are introduced and the variables $\psi_H, \theta_H, \phi_H, \theta', \phi', h$ are used to represent their first order secular parts. The associated equations for the secular motions (5.11) under the influence of dissipative aerodynamic torque become

$$\dot{\psi}_H = -(c_{\theta o} + s_{\theta o} \cot_{\theta_H} c_{\psi_H}) \dot{\Omega}$$

$$- \frac{c\rho_a}{4 s_{\theta_H}} \left(\frac{\mu}{p^*}\right)^{1/2} [\frac{1}{A} C_{32} k_8 s^2_{\theta'} \; c_{\theta'} + \frac{1}{C} C_{23} k_8 (1 - c^2_{\theta'}) c_{\theta'}] \quad \text{(a)}$$

$$\dot{\theta}_H = - s_{\theta o} \; s_{\psi_H} \dot{\Omega} \tag{6.9}$$

$$+ \frac{c\rho_a}{2} \left(\frac{\mu}{p^*}\right)^{1/2} C_{32} k_9 [\frac{1}{2A} s^2_{\theta'} - \frac{1}{C}(1 - \frac{1}{2} s^2_{\theta'})] c_{\theta_H} c_{\theta'} \quad \text{(b)}$$

$$\dot{\phi}_H = \frac{h}{A} + \frac{s_{\theta o} \; c_{\psi_H}}{s_{\theta_H}} \dot{\Omega}$$

$$- \frac{c\rho_a}{4} \left(\frac{\mu}{p^*}\right)^{1/2} k_8 [-\frac{1}{A} C_{32} s^2_{\theta'} c_{\theta'} + \frac{1}{C}(1 - c^2_{\theta'}) c_{\theta'}] \cot_{\theta_H} \quad \text{(c)}$$

$$\dot{\theta}' = \frac{c\rho_a}{2}\left(\frac{\mu}{p^*}\right)^{1/2}[k_0(\frac{C_{33}}{A} - \frac{C_{11}}{C})\, s_{\theta'}\, c_{\theta'}$$

$$+ k_9\, (\frac{C_{32}}{A}\, s^2_{\theta'} - \frac{C_{33}}{C}\, c^2_{\theta'})\, s_{\theta_H}\, s_{\theta'}] \quad \text{(d)}$$

$$\dot{\phi}' = -\frac{h}{A}(1- \frac{A}{C})\, c_{\theta'} \quad \text{(e)} \quad (6.9)$$

$$\dot{h} = \frac{1}{2}\, h\, c\rho_a\left(\frac{\mu}{p^*}\right)^{1/2}[- k_8(\frac{C_{11}}{A}\, s^2_{\theta'} + \frac{C_{33}}{C}\, c^2_{\theta'})$$

$$- k_9(\frac{C_{32}}{A} + \frac{C_{23}}{C})\, s_{\theta_H}\, s^2_{\theta'}\, c_{\theta'}] \quad \text{(f)}$$

It may be seen from equations (6.9) that all six variables have secular, long-term variations due to the presence of the dissipative aerodynamic torque, $\underline{N}_{AD}$. The terms which involve k_0 are more important than those containing k_8 and k_9 since k_8 and k_9 are smaller quantities than k_0.

The problem associated with a circular orbit and constant air density can be obtained readily by letting $e = 0$ and ρ_a = constant, so that $k_8 = k_9 = 0$, $k_0 = \rho_a$. The differential equations for this case are

$$\dot{\psi}_H = -(c_{\theta\circ} + s_{\theta\circ}\cot_{\theta_H} c_{\psi_H})\, \dot{\Omega} \quad \text{(a)}$$

$$(6.10)$$

$$\dot{\theta}_H = - s_{\theta\circ}\, s_{\psi_H}\, \dot{\Omega} \quad \text{(b)}$$

$$\dot{\phi}_H = \frac{h}{A} + \frac{s_{\theta^\circ}\, s_{\psi_H}}{s_{\theta_H}} \dot{\Omega} \qquad (c)$$

$$\dot{\theta}' = \frac{c\rho_a V_o}{2} \left(-\frac{C_{11}}{A} + \frac{C_{33}}{C}\right) s_{\theta'}\, c_{\theta'} \qquad (d) \qquad (6.10)$$

$$\dot{\phi}' = -\frac{h}{A}\left(1 - \frac{A}{C}\right) c_{\theta'} \qquad (e)$$

$$\dot{h} = -\frac{c\rho_a V_o}{2}\left(\frac{C_{11}}{A} s^2_{\theta'} + \frac{C_{33}}{C} c^2_{\theta'}\right) h \qquad (f)$$

Equations (6.10(d)) and (6.10(f)) can be integrated directly. It is found, from (6.10), that

$$\tan_{\theta'} = \tan_{\theta'_o} e^{N_o t} \qquad (6.11)$$

and from (6.10(f)), that

$$h = h_o \exp \left\{ -\frac{c\rho_a V_o}{2} \left[\left(\frac{C_{11}}{A} + \frac{C_{33}}{C}\right) t \right.\right.$$

$$\left. + \ln \left[\frac{\left(\tan^2_{\theta'_o} + e^{-2N_o t}\right)^{\frac{C_{11}}{2N_o A}}}{\left(1 + \tan^2_{\theta'_o} e^{2N_o t}\right)^{\frac{C_{33}}{2N_o C}}} \right] \qquad (6.12)$$

where

$$N_o = \frac{c\rho_a V_o}{2}\left(\frac{C_{33}}{C} - \frac{C_{11}}{A}\right) \qquad (6.13)$$

and θ'_o and h_o are the initial values of θ' and h, respectively.

From equations (6.11) and (6.12), it can be seen that, under the influence of dissipative aerodynamic torque, the magnitude of the angular momentum vector decreases exponentially, approaching zero in the limit. However, the angle between the x'-axis and the angular momentum vector may either decrease or increase with time, approaching either zero or $\pi/2$ as a limiting value, respectively, depending on whether N_o is negative or positive. If C_{11} and C_{33} are of the same order of magnitude, equation (6.11) indicates that the body will eventually spin about the axis of the maximum moment of inertia. In case θ'_o is zero, θ' will remain zero and hence θ' is a constant of motion.

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